Visualizing the roots of a cubic equation

Páll-Szabó Ágnes Orsolya
pallszaboagnes@math.ubbcluj.ro
Babeş-Bolyai University, Cluj-Napoca
Romania

Abstract

In this article we discuss some methods of visualizing complex roots of a cubic equation, with examples. First we present a brief history of cubic equations and define the roots of general cubic equation. The first method presented is De Moivre's method, folloved by geometric interpretation and graphic determination of the roots and other more or less known methods.

1 Introduction

In this article we discuss a few methods of visualizing the complex roots of cubic equations, with examples. Visualization is very important, because, as we shall see, abstract concepts and symbols which are difficult to understand can be presented much easier. Even at school students fail in learning mathematics because teachers do not pay attention to the basic concepts and use formulas without explaining where they came from and will not try to present them in a visual way, using computers and computer softwares.

Another outcome of this article is the proof of Bardell's observation which stated that in the case of a general cubic equation $az^3+bz^2+cz+d=0$ if the coefficients satisfy the $b^2-3ac=0$ equation then the $z_1,z_2,z_3\in\mathbb{C}$ roots form an equilateral triangle (see theorem 2 and it's proof). For generating the images we used the software packages Geogebra and Matlab.

2 History of the different solutions

Cubic equations have been known since ancient times. One of the simplest and oldest studied cubic equations is the one connected with the problem of doubling the cube. According to the legend the citizens of the ancient Delos consulted the oracle of Delphi in order to solve their problems. The answer of the oracle was that they must double the volume of the altar of Apollo, which was a cube. Although the task seems simple, it has been proven impossible to fulfill with a compass and a straightedge.

At the beginning of the 16th century, the Italian mathematician Scipione del Ferro found a method for solving the $x^3+mx=n$ class of cubic equations. All cubic equations can be reduced to this form if we allow m and n to be negative, but negative numbers were not known to him at that time. Before his death Del Ferro told his student Antonio Fiore about his method. In 1530, Niccolo Tartaglia worked out a general method to solve the $x^3 + mx = n$ equation. His results were published by Gerolamo Cardano in 1545.

François Vieta independently derived the trigonometric solution for the cubic with three real roots, his results were expanded by Descartes.

3 Algebraic methods

3.1 Zero points of a cubic function

The general cubic equation has the form $ax^3 + bx^2 + cx + d = 0, a \neq 0$. As a consequence of the intermediate value theorem (according to Bolzano's theorem if a continuous function has values of opposite signs within an interval, then it has a root in that interval) every cubic equation with real coefficients has at least one real solution. We have three possible cases using the discriminant,

$$\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2.$$

- If $\Delta > 0$, then the equation has three distinct real roots.
- If $\Delta = 0$, then the equation has a multiple real root.
- If $\Delta < 0$, then the equation has one real root and two complex conjugate roots.

For finding the general formula of the roots we consider the following:

$$u_1 = 1, u_2 = \frac{-1 + i\sqrt{3}}{2}, u_3 = \frac{-1 - i\sqrt{3}}{2}$$

which are the three cube roots of unity.
We calculate

$$\Delta_0 = b^2 - 3ac, \Delta_1 = 2b^3 - 9abc + 27a^2d \tag{1}$$

and

$$C = \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}. \tag{2}$$

The general formula for the roots is:

$$x_k = -\frac{1}{3a}\left(b \,+\, u_kC \,+\, \frac{\Delta_0}{u_kC}\right), \qquad k \in \{1, 2, 3\}. \tag{3}$$

There are other methods to calculate the roots of a cubic equation, like Cardano's method or Lagrange's method.

3.2 De Moivre's method

According to Bardell [4], a general cubic $az^3 + bz^2 + cz + d = 0$ with real or generally complex coefficients may be expressed in the vertex form $a(z+h)^3 + k = 0$ where $h = \dfrac{b}{3a}, k = \dfrac{27da^2 - b}{27a^2}$, if the coefficients satisfy the $b^2 - 3ac = 0$ condition. He observed that "Only in those cases when the cubic reduces to the vertex form do the roots form an equilateral triangle and lie equally spaced around de Moivre's circle. This observation applies whether the coefficients are real or complex."

To prove his conjecture we use the following theorem [1]:

Theorem 1 *Let $z_1, z_2, z_3 \in \mathbb{C}$ be the vertices of an equilateral triangle if*

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_1z_3 + z_2z_3. \tag{4}$$

Theorem 2 *(Main result) Let $az^3 + bz^2 + cz + d = 0$ be a general cubic. If the coefficients satisfy the $b^2 - 3ac = 0$ equation then the $z_1, z_2, z_3 \in \mathbb{C}$ roots form an equilateral triangle.*

Proof.
If z_1 is the root of the cubic then $az_1^3 + bz_1^2 + cz_1 + d = 0$
If z_2 is the root of the cubic then $az_2^3 + bz_2^2 + cz_2 + d = 0$
If z_3 is the root of the cubic then $az_3^3 + bz_3^2 + cz_3 + d = 0$.
Adding these three equations we get:

$$a(z_1^3 + z_2^3 + z_3^3) + b(z_1^2 + z_2^2 + z_3^2) + c(z_1 + z_2 + z_3) + 3d = 0. \tag{5}$$

We apply the Vieta's formulas for our cubic:

$$z_1 + z_2 + z_3 = -\frac{b}{a}, \quad z_1z_2 + z_1z_3 + z_2z_3 = \frac{c}{a}, \quad z_1z_2z_3 = -\frac{d}{a}.$$

Substituting the Vieta's formulas in the equation (5) we get

$$z_1^3 + z_2^3 + z_3^3 = -\frac{3d}{a}. \tag{6}$$

If we raise the first equation from Vieta's formulas to the third power and expand, we get:

$$z_1^3 + z_2^3 + z_3^3 + z_1z_2(z_1 + z_2) + z_1z_3(z_1 + z_3) + z_2z_3(z_2 + z_3) = -\frac{b^3}{3a^3}. \tag{7}$$

From Vieta's formulas, equation (6) and equation (7) follows that

$$\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = -\frac{b^2}{3ad}. \tag{8}$$

This is equivalent to

$$b^2 = 3ac \text{ or } \Delta_0 = b^2 - 3ac. \tag{9}$$

and the (2) equation will be $C = \sqrt[3]{\Delta_1}$. ■

De Moivre's theorem states that [5]: if p and q are integers prime to each other, then

$$[r(cos\theta + isin\theta)]^{\frac{p}{q}} = r^{\frac{p}{q}}[cos((\theta + 2\pi k)\frac{p}{q}) + isin((\theta + 2\pi k)\frac{p}{q})], k = 0, 1, 2, \ldots, q-1.$$

This is useful for finding the $\frac{p}{q}-$th root of a complex number.

Example: The $iz^3 + (1-i)z^2 - \frac{2}{3}z + 3 - 4i = 0$ cubic can be expressed in vertex form since the condition $b^2 - 3ac = 0$ is satisfied with $(1-i)^2 - 3i(-\frac{2}{3}) = 0$. Then $h = -\frac{i+1}{3}, k = \frac{82-109i}{27}$ and the vertex form is $i(z - \frac{i+1}{3})^3 + \frac{82-109i}{27} = 0$.

After rearrangement $(z - \frac{i+1}{3})^3 = \frac{109+82i}{27}$. The problem is reduced to finding the cube roots of $\frac{109+82i}{27}$ which by de Moivre's method are $2.0084+0.6862i$, $-0.8097+1.6075i$, $-0.1986-1.2937i$. Indeed the roots form an equilateral triangle.

4 Geometric interpretation of the roots

4.1 Three real roots

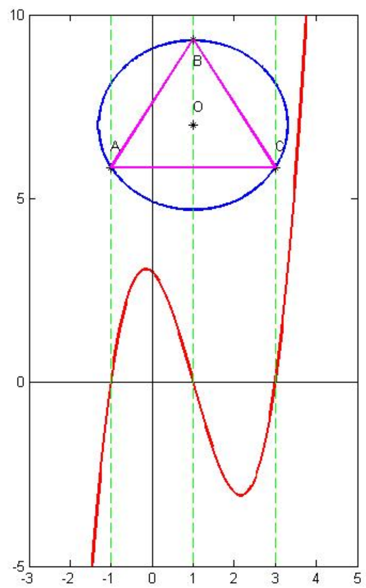


Figure 1: $f(x) = x^3 - 3x^2 - x + 3$

The three real roots of any cubic are the projections on the x-axis of the vertices A, B and C of an equilateral triangle. The center of the equilateral triangle has the same abscissa as the inflection point.

Let $ax^3 + bx^2 + cx + d = 0$. If $a \neq 1$, we divide the equation by the leading coefficient.

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0.$$

After we do variable exchange $x = t - \dfrac{b}{3a}$, we get the so-called "incomplete" or depressed cubic:

$$t^3 + pt + q = 0. \tag{10}$$

Graphically this corresponds to shifting the graph horizontally. This shift moves the point of inflection and the centre of the circle onto the y-axis.

The three vertices of the equilateral triangle are the three cube roots of a complex number $z = x + iy$. From de Moivre's formula we get:

$$z^3 = (r(cos(\alpha) + i \cdot sin(\alpha)))^3 = r^3(cos(3\alpha) + i \cdot sin(3\alpha))$$

$$z^3 = (x + i \cdot y)^3 = (x^3 - 3xy^2) - i \cdot (y^3 - 3yx^2) = r^3(cos(3\alpha) + i \cdot sin(3\alpha))$$

Since we are projecting onto the x-axis, we will need only the real part of the equation:

$$x^3 - 3xy^2 = r^3 cos(3\alpha). \tag{11}$$

The equation of the circle is:

$$x^2 + y^2 = r^2$$

or equivallently

$$y^2 = r^2 - x^2. \tag{12}$$

We substitute (12) in (11) and dividing by 4, we get:

$$x^3 - 3x\frac{r^2}{4} - \frac{r^3}{4}cos(3\alpha) = 0. \tag{13}$$

We set the coefficients of the equations (10) and (13) equal, the result is:

$$p = -\frac{3r^2}{4}, q = -\frac{r^3 cos(3\alpha)}{4}.$$

First we solve the equation for $r : r^2 = -\dfrac{4p}{3}$, $r = \pm\sqrt{\dfrac{-4p}{3}}$, this way $p < 0$.

Then we solve it for $\alpha : cos(3\alpha) = \dfrac{-4q}{r^3}$, $\alpha = \dfrac{1}{3}arccos(\dfrac{-4q}{r^3})$, so $\left|\dfrac{4q}{r^3}\right| \leq 1$.

The three real roots of the cubic (10) that generate the vertices of an equilateral triangle are:

$$\begin{cases} z_1 = rcos(\alpha) \\ z_2 = rcos(\alpha + \dfrac{2\pi}{3}) \\ z_3 = rcos(\alpha - \dfrac{2\pi}{3}) \end{cases}$$

For more information and for its application see [3], [15].

4.2 One real and two complex roots

4.2.1 In the Cartesian plane

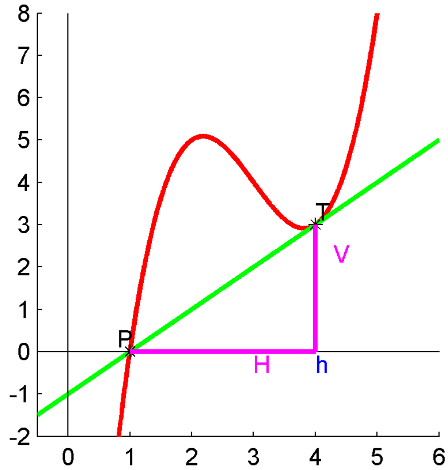


Figure 2: $f(x) = x^3 - 9x^2 + 25x - 17$

If a cubic equation is plotted in the Cartesian plane, the real root can be seen graphically as the x-intercept of the curve. If the complex conjugate roots are written as $h \pm ik$, then h is the abscissa of the tangency point of a line that is tangent to the cubic curve and intersects the x-axis at the exact point as does the cubic curve, and k^2 is the slope of the tangent of the angle between this line and the x-axis.
Let

$$y = ax^3 + bx^2 + cx + d \tag{14}$$

be a general cubic equation with one real root at p and two complex roots at $h \pm ik$. The curve cuts the x-axis at $P(p, 0)$. We draw the tangent line in $T(h, k^2(h-p))$ to the cubic which goes through the real root $P(p, 0)$.

$$\frac{dy}{dx} = 3ax^2 + 2bx + c.$$

Since p and $h \pm ik$ are the roots of the cubic

$$ax^3 + bx^2 + cx + d = 0, \tag{15}$$

from Vieta's formulas we have :

$$b = -a(p + 2h); c = a(2hp + h^2 + k^2); d = -ap(h^2 + k^2). \tag{16}$$

Putting $x = h$ and substituting these values in (14) we get $y = ak^2(h - p)$. The tangent at the point (x_1, y_1) is :

$$y - y_1 = (3ax_1^2 + 2bx_1 + c)(x - x_1). \tag{17}$$

With $x_1 = h, y_1 = ak^2(h-p)$ and substituting (16) in (17) we get $y = ak^2(x-p)$ as the tangent at $T(h, k^2(h-p))$ passes through $P(p, 0)$. For the details see [7], [8].

Example: We consider $f(x) = x^3 - 9x^2 + 25x - 17$ and we draw the graph of the function. We find the line which goes through the real root $P(1, 0)$ and which is also tangent to the graph of the function (the green line) in $T(4, 3)$. Then $k = \sqrt{\frac{V}{H}} = 1$, $H = h - p = 3$, $V = k^2(h-p) = 3$, $h \pm ik = 4 \pm i$ which are the complex roots of the cubic.

4.2.2 In the Argand plane

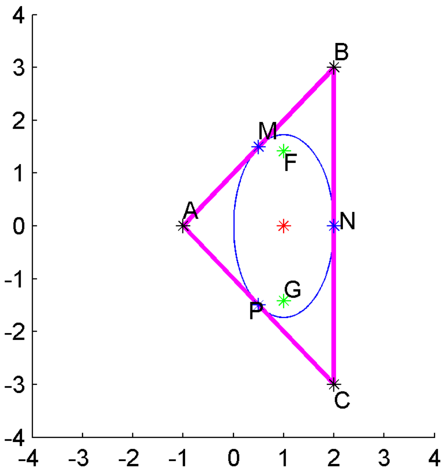
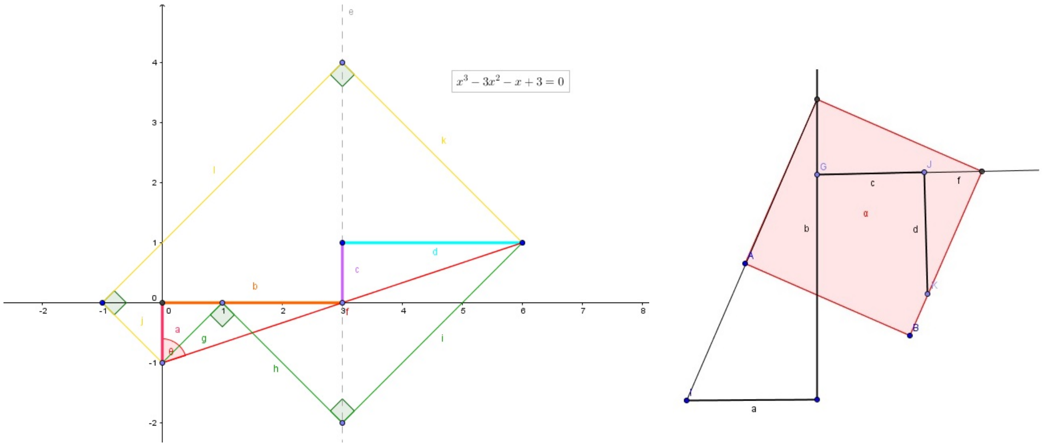


Figure 3: $p(x) = x^3 - 3x^2 + 9x + 13$, roots at $-1, 2+3i, 2-3i$

Dan Kalman [12], [13] presented Marden's theorem in the following form:
" Let $p(z)$ be a third-degree polynomial with complex coefficients, and whose roots z_1, z_2, and z_3 are noncollinear points in the complex plane. Let T be the triangle with vertices at z_1, z_2, and z_3. There is a unique ellipse inscribed in T and tangent to the sides at their midpoints. The foci of this ellipse are the roots of $p'(z)$."

The zeroes of $p(x)$ are the vertices A, B and C of an isosceles triangle (the triangle is isosceles because the real root is on the x-axis and the two complex conjugates roots appear symmetrically above and below the real axis). The zeroes of $p'(x)$ are the foci of the Steiner inellipse i. e. the unique ellipse tangent to the three sides of a triangle at their midpoints. The zero of $p''(x)$ is the center point of the ellipse and the centroid of the triangle. Marden's theorem states that the green dots (F, G) in Figure 3 are the foci of the ellipse. The major axis and the foci of the ellipse lie on the real axis if the angle at the vertex on the real axis is smaller than $\frac{\pi}{3}$. The major axis is vertical and its foci are complex if the angle is greater than $\frac{\pi}{3}$. If the angle is equal to $\frac{\pi}{3}$, the triangle is equilateral, the Steiner inellipse is the triangle's incircle. See, for example, [16].

4.3 Lill's method



(a) $f(x) = x^3 - 3x^2 - x + 3$, roots at $-1, 1, 3$ (b) paper folding

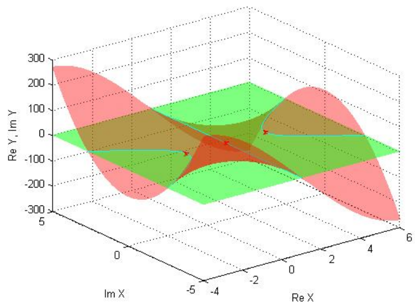
Figure 4: Lill's method

Lill's method is a geometric approach for finding roots of polynomials with real coefficients, that can be implemented with computer graphics, or ordinary graph paper. The method was discovered by an Austrian engineer, Eduard Lill, and presented at the Paris Exposition, in 1867. With this method we can find real solutions, and it will work for equations of any degree. It is a way of visualizing the roots geometrically. See [6], [18]. We consider a general cubic equation:

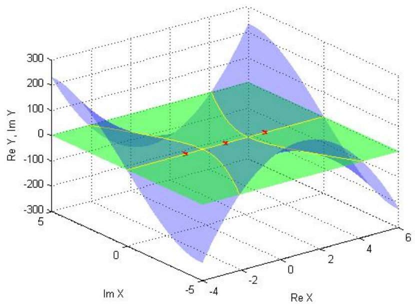
$$ax^3 + bx^2 + cx + d = 0.$$

If $a \neq 1$, we divide the equation by the leading coefficient. From the starting point $(0, -1)$ we lay out counter-clockwise perpendicular segments with lengths equal to the a, b, c and d coefficients. If we have a negative coefficient we go upwards on the vertical path, and downwards if the coefficient is positive. On the horizontal path we go left if the coefficient is positive, and right in the case of a negative one. We construct the right angle pathway that reflects off the walls from the starting point to the terminus (final point). One solution of the equation is given by the negative of the slope that is $x = -tan\ \theta$ where $\theta < \pm 90°$. For example, the yellow right-angle subpath reflects off of the b-line at $x = -1$, providing that solution to the cubic equation.
Italian mathematician Margherita P. Beloch (1936) showed how Lill's method could be adapted to solve cubic equations using paper folding (origami). See [11].

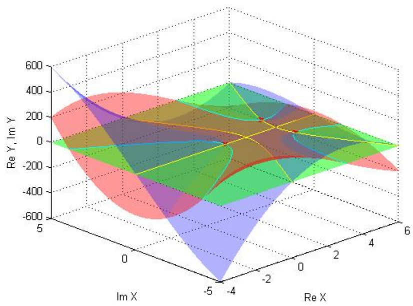
4.4 Surface plot



(a) Surface A (red) and Re y=0 (green)



(b) Surface B (blue) and Re y=0 (green)



(c)

Figure 5: $f(x) = x^3 - 9x^2 + 25x - 17$, roots at $1, 4+i, 4-i$, figure generated with Matlab

We consider $y = ax^3 + bx^2 + cx + d$ and $X = G + iH$ a complex value. If we substitute it in the equation the result is

$$y = a(G + iH)^3 + b(G + iH)^2 + c(G + iH) + d,$$

$$y = (aG^3 - 3aGH^2 + bG^2 - bH^2 + cG + d) + i(3aG^2H - aH^3 + 2bGH + cH) = A + iB,$$

with $A = aG^3 - 3aGH^2 + bG^2 - bH^2 + cG + d$ and $B = 3aG^2H - aH^3 + 2bGH + cH$, where $Re(X) = G, Im(X) = H, Re(y) = A, Im(y) = B$. Then we plot the surfaces. The complex roots are where $A = B = 0$ or $Re(y) = Im(y) = 0$. This is illustrated in Figure 5: we have three points where surfaces A and B intersect each other and simultaneously intersect a horizontal plane positioned at zero altitude. See [17].

Matlab code for Figure 5:

```
g = -4:0.1:6; h =-5:0.1:5; t= -4:0.1:6; tt = -5:0.1:5; x=-5:0.1:6; z=0;
f=x.^3-3*x.^2-x+3;
[g,h] = meshgrid(g,h);
a=g.^3-3*g.*h.^2-3*g.^2+3*h.^2-g+3;
b=3*h.*g.^2-h.^3-6*g.*h-h;
[Xmesh, Ymesh,Zmesh] = meshgrid(t,tt,z);
surf(g,h,a,'FaceColor','red','FaceAlpha',0.4,'EdgeColor','none');
```

```
hold on
xlabel('Re X');ylabel('Im X');zlabel('Re Y, Im Y');
surf(g,h,b,'FaceColor','blue','FaceAlpha',0.3,'EdgeColor','none');
mesh(Xmesh,Ymesh,Zmesh,'FaceColor','green',...
        'FaceAlpha',0.5,'EdgeColor','none');
scatter3(1,0,0,'red','X','LineWidth',2);
scatter3(-1,0,0,'red','X','LineWidth',2);
scatter3(3,0,0,'red','X','LineWidth',2);
contour3(g,h,b,[0 0],'y');
hold off
```

4.5 Cubic equation with complex coefficients

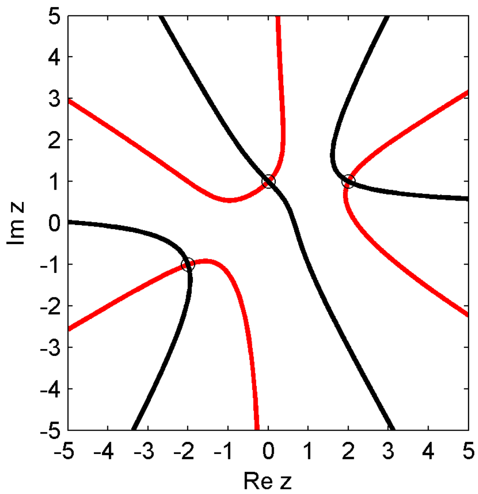


Figure 6: $f(z) = z^3 - iz^2 - (3+4i)z - 4 + 3i$, roots at $i, 2+i, -2-i$

When it comes to finding zeros, the complex function $f(z)$ has an advantage: there are two equations (real and imaginary parts of f) for two unknowns (real and imaginary parts of z).
Example: Regarding $f(z) = z^3 - iz^2 - (3+4i)z - 4 + 3i$, the complex roots are : $z_1 = i, z_2 = 2+i, z_3 = -2-i$. If we substitute $z = x + iy$ in the equation $f(z) = 0$ and we expand it, we get:

$$(x^3 - 3xy^2 + 2xy - 3x + 4y - 4) + i(3x^2y - y^3 - x^2 + y^2 - 3y - 4x + 3) = 0.$$

Separating the real and imaginary parts, and setting each one equal to zero, we have two real equations:

$$x^3 - 3xy^2 + 2xy - 3x + 4y - 4 = 0 \text{ and } 3x^2y - y^3 - x^2 + y^2 - 3y - 4x + 3 = 0. \qquad (18)$$

Then we plot y depending on x and x with respect to y (y as a function of x and x as a function of y). This is illustrated in Figure 6. The cubic's complex roots are at the intersection points of the two graphs. See [9].

4.6 Modulus surface

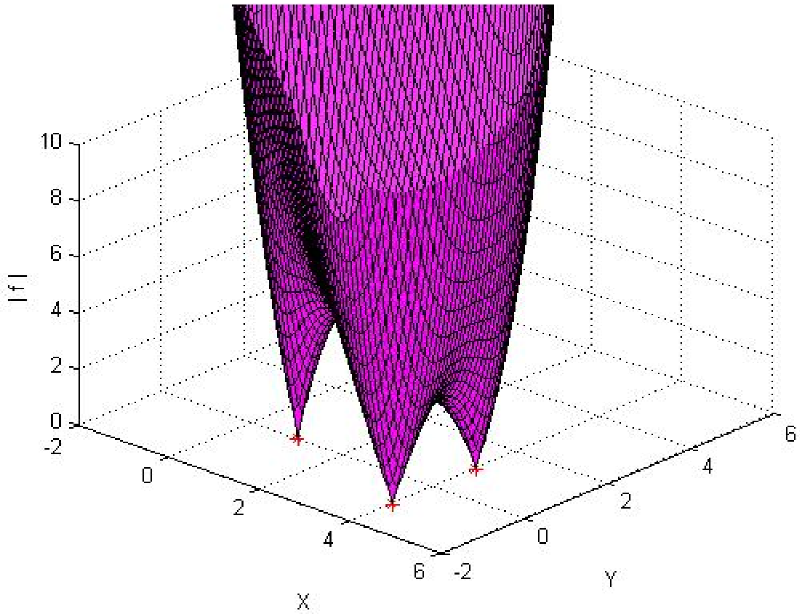


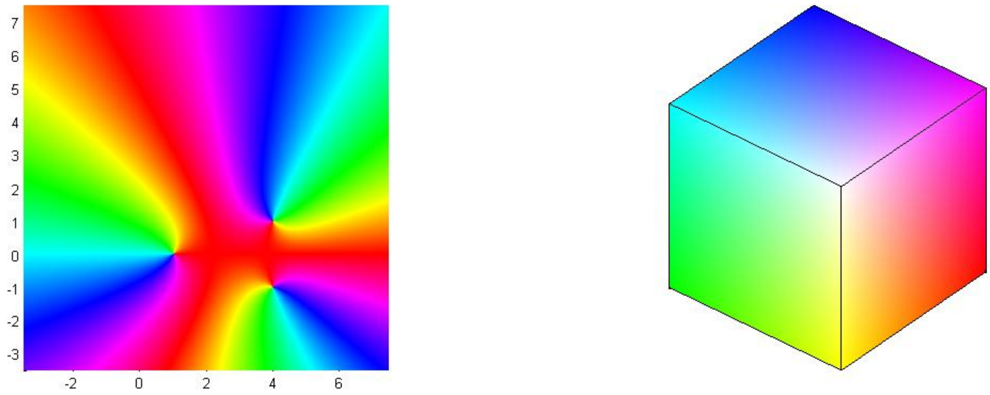
Figure 7: $f(z) = z^3 - 9z^2 + 25z - 17$, roots at $1, 4+i, 4-i$

Matlab code for Figure 7:

```
x=-2:0.1:6;
y = x;
[x,y] = meshgrid(x,y);
z = x + i*y;
f = z.^3-9*z.^2+25*z-17;
surf(x,y,abs(f),'FaceColor','magenta','FaceAlpha',0.8,'EdgeColor','k');
hold on
zlim([0 10]);xlabel('X');ylabel('Y');zlabel('|f|');
plot3(1,0,0,'r*');plot3(4,1,0,'r*');plot3(4,-1,0,'r*');
hold off
```

If $w = x+iy, w \in \mathbb{C}, x, y \in \mathbb{R}$ then the modulus is defined as $|w| = \sqrt{x^2+y^2} \geq 0$. The modulus surface of the function $f(z) = az^3+bz^2+cz+d$ is the surface defined by three real variables x, y and $|f(z)|$. The graph always lies on or above the Argand plane. $|f(z)| = 0$ if and only if $f(z) = 0$, so the modulus surface touches the complex plane in those particular points where the function is zero. Complex roots are found where the modulus surface has minimum points touching the xy-plane. The cubic function $f(z) = z^3 - 9z^2 + 25z - 17$ has the modulus surface as seen in Figure 7, where the roots of the polynomial are $1, 4 \pm i$ and the surface has exactly those three points that touch the xy-plane. This method can also be used with complex coefficient polynomials and with polynomials that have higher degree. This method enables us to see how the zeros of a polynomial move when a single coefficient is changed(surfaces became easy to draw with the use of computer graphics). This method is presented in [2], [17] and [14].

4.7 Color representation

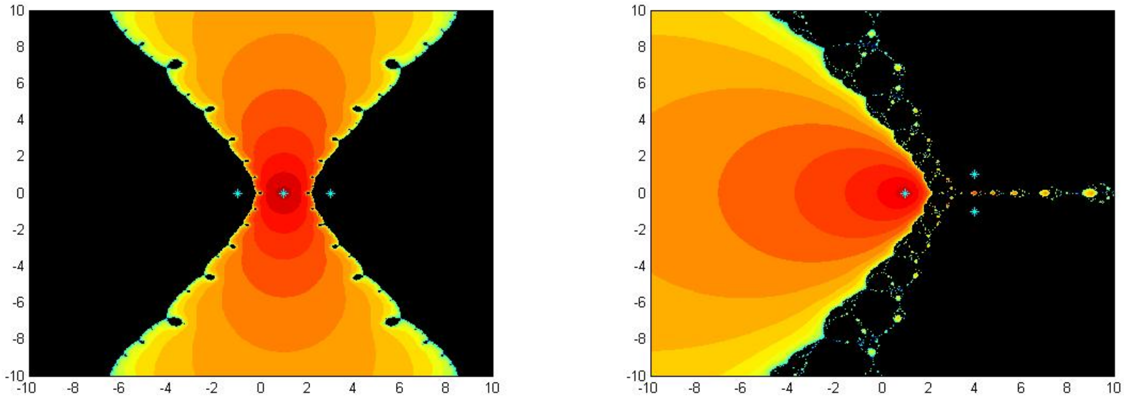


(a) $f(x) = x^3 - 9x^2 + 25x - 17$, roots at $1, 4 + i, 4 - i$

(b) RGB cube (red-green-blue)

A graph of a complex function of one complex variable with the complex plane itself, two-dimensional as well, is an object in four real dimensions. That's why complex functions are difficult to visualize in a three-dimensional space. With domain coloring we represent four-dimensional objects in two dimensions. This method sets a one-to-one map between the RGB cube (a 3-D geometrical model of the color additive system, a three-vectors space where each component is a color among Red, Green, Blue) and the complex plane, so that each complex point matches only one color. With the color wheel method we graphically represent complex functions.The color wheel method assigns a color to the point of a complex plane.The domain coloring technique gives a continuous and bijective map from the complex plane to the color wheel: 0 maps to white, 1 maps to red, -1 maps to cyan, and the remaining six roots of unity map to yellow, green, blue, violet and infinity maps to black. The phase (argument) is encoded as hue, while lightness corresponds to the modulus. The behavior and the basic properties (zeros and their multiplicities, periodicity, symmetry and range) of the function can be rapidly overviewed with this method. See [10],[17] ,[19].

5 Approximate solution: Newton's method



(c) $f(x) = x^3 - 3x^2 - x + 3$

(d) $f(x) = x^3 - 9x^2 + 25x - 17$

Figure 8: Basin of attraction with Newton's method

Newton's method is an efficient way to approximate solutions of equations of the form $f(x) = 0$. Newton's method involves choosing an initial point x_0 and then finding a sequence of numbers $x_0, x_1, x_2, \cdots$ that converge to a solution. Select a point x_0 , arbitrarily close to the function's root. Consider the tangent line to the function f at the point $(x_0, f(x_0))$ than we can compute the x-intercept of this tangent line at $x = x_1$. Repetition of this algorithm generates a sequence of x values $x_0, x_1, x_2, \cdots$ by the rule

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, f'(x_n) \neq 0.$$

Consider $f(x) = x^3 - 9x^2 + 25x - 17$. It has three solutions at $x = 1, x = 4 + i$ and $x = 4 - i$. In the table below we can see that for different starting values the Newton method converges to the roots.

10.0000 - 0.5000i	11.0000 + 5.0000i	1.0000 + 0.1000i
7.7035 - 0.3326i	8.3667 + 3.3216i	1.0060 + 0.0005i
6.1714 - 0.2234i	6.6230 + 2.2065i	1.0000 - 0.0000i
5.1238 - 0.1581i	5.4721 + 1.4807i	1.0000 - 0.0000i
4.3201 - 0.1407i	4.7112 + 1.0472i	1.0000 - 0.0000i
3.2695 - 0.3676i	4.2113 + 0.8809i	
4.7861 - 0.4576i	3.9867 + 0.9618i	
4.0767 - 0.4735i	4.0002 + 1.0011i	
3.7045 - 1.1505i	4.0000 + 1.0000i	
3.9538 - 0.9501i	4.0000 + 1.0000i	
4.0029 - 1.0016i		
4.0000 - 1.0000i		
4.0000 - 1.0000i		

The basin of attraction of root r is the set of all x_0 numbers, such that Newton's method starting at x_0 converges to r. For example, if we pick any complex point in the yellow regions in Figure 9 (b), when it is used as the initial point in the Newton's method, it will converge to the root 1. Every starting value that converges to 1 is in the basin of attraction of its own. All the points converge very quickly to their respective roots, so we can see in the picture the color shading.
In Figure 9 (a) $f(x) = x^3 - 3x^2 - x + 3$ is pictured, where the roots are $x = 0, \pm 1$. We indicate by different color shades the amount of iterations needed to converge to that particular point.

6 Conclusion

As we have seen above, even a task that can be difficult to deal with, like computing the roots of a cubic equation can be understandable with the use of different visualization methods, because - a picture is worth more than a thousand words.

Using some of these methods in the classroom can attract the students' attention and can help them understand the problems and solutions more easily. Unfortunately, the use of computers in teaching mathematics has its limits: in some regions of the world the technological background is missing, in other cases teachers do not have the knowledge needed to use such methods.

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